The perspective projection of the Earth onto a tilted plane is useful in simulating photographs taken from Earth orbiting space vehicles. A new approach to the calculation of projection coordinates is demonstrated by using vector geometry.

The 'Tilted Camera' perspective projection of the Earth

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INTRODUCTION

Space photographs offer spectacular views of the Earth, often showing the curved horizon against the backdrop of space and the continental shapes contrasted by the oceans. Such photographs taken from Earth orbiting satellites are perspective projections onto the focal plane of the camera. The perspective point or focal point of the camera, is a point in space somewhere between the surface of the Earth and infinity, and the camera focal plane is usually obliquely inclined to the line joining the focal point and the centre of the Earth.

Space photographs can be simulated by a perspective projection of the meridians, parallels, and the continental outlines onto an oblique plane situated at some point in space between the perspective point and the earth. The perspective point can be considered as the focal point of the lens system of an imaginary space camera, and the projection plane is analogous to a camera focal plane, except it lies between the focal point and the object, rather than behind the focal point as in a real camera.

Trigonometric equations for perspective projections of the Earth, simulating space photographs, were developed by Snyder (1981). The technique developed in the following pages makes use of vector geometry to achieve these projections.

NOTATION AND CONVENTIONS

Figure 1 shows a section of the spherical Earth and a perspective point F at some position in space. Points on the Earth's surface are to be projected geometrically onto a plane situated somewhere between F and the surface of the Earth.

The XYZ Cartesian system is shown with the origin O at the centre of the spherical Earth, the positive Z axis directed through the North Pole, the positive X axis directed through the intersection of the Greenwich meridian and the Equator, and the positive Y axis advanced 90° eastwards from the X axis in the plane of the Equator.

The following notation applies:

ϕ is the latitude, reckoned positive north and negative south of the Equator from 0° to 90°.

λ is the longitude, reckoned positive east and negative west of the Greenwich meridian from 0° to 180°.

R is the radius of the spherical Earth.

Several relationships between the satellite camera location and the orientation of the projection plane need to be explained:

(a) The satellite camera perspective point is at F. The straight line OF intersects the surface of the Earth at G, the satellite's ground point. The distance FG is the height H of the satellite above the earth.

(b) The axis of the camera is the straight line through the lens system and perpendicular to the projection plane.

(c) The axis of the camera is pointed at C on the surface of the Earth. The line FC intersects the projection plane at C′ which is the centre of the image and the origin of the X′ Y′ projection plane coordinate system. Thus the angles FC′ Y′ and FC′ X′ are both right angles.

(d) The Y′ axis of the projection plane lies in the plane OCF.

(e) The distance FC′ is analogous to the focal length of a camera.

(f) The line FP intersects the projection plane at P′; hence P′ is the geometric projection of P.

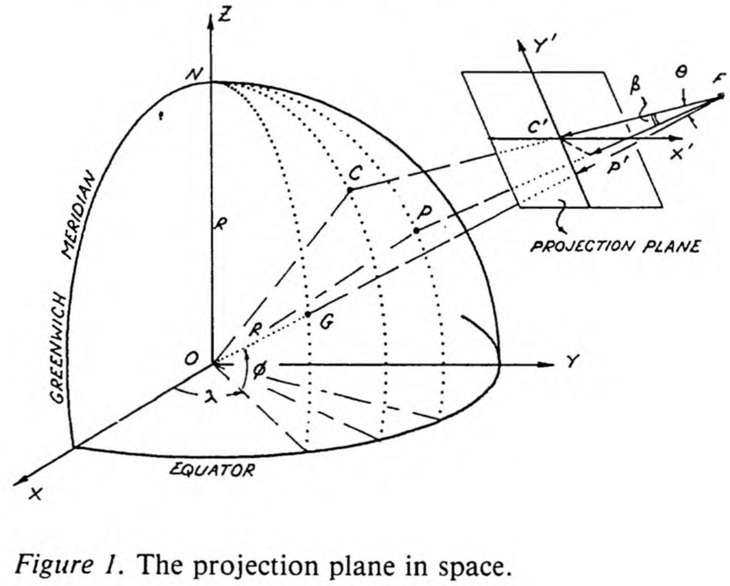


Figure 1. The projection plane in space.

To simulate a space photograph, the Cartesian coordinates of P′ on the projection plane must be calculated from a knowledge of the location and orientation of the imaginary satellite camera and the geographical location of point P on the Earth's surface.

A VECTOR SOLUTION OF THE PROBLEM

A method for the solution of the coordinates of P′ on the projection plane may be obtained by using vector geometry. *Figure 2* shows the projection plane and the necessary geometric relationships to develop this method.

The solution depends on a knowledge of the following elements:

(i) The geographical location (ϕ_G, λ_G) of the camera's ground point.
(ii) The geographical location (ϕ_C, λ_C) of the point on the Earth's surface at which the camera is pointing.
(iii) The distance FG, which is the camera's height H, above the Earth's surface.
(iv) The distance FC′ between the focal point and the projection plane.
(v) The geographical location (ϕ_P, λ_P) of any other point P on the Earth.

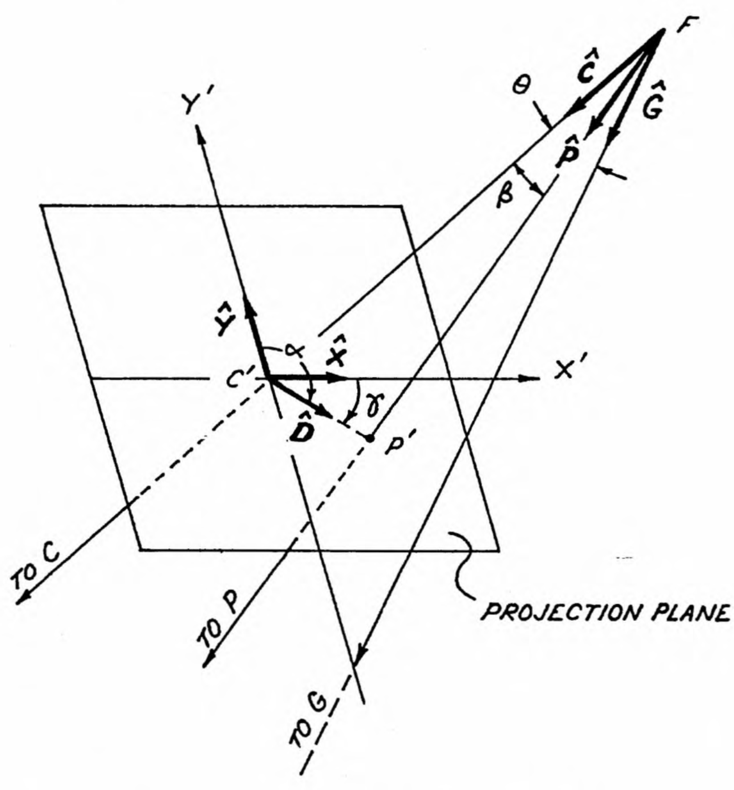


Figure 2. Vector diagram and the projection plane.

1. *Calculation of Cartesian coordinates (X, Y, Z)*
The 3-dimensional Cartesian coordinates of points G, C, F, and P can be calculated from their geographical coordinates (ϕ, λ) by the formulae:

$$X = (R + H)\cos\phi \cos\lambda$$
$$Y = (R + H)\cos\phi \sin\lambda$$
$$Z = (R + H)\sin\phi \qquad (1)$$

where

H is the height above the spherical Earth.

For points G, C, and P on the surface of the Earth, H is generally considered to be zero, whilst for point F in space, H is the height of the camera above the Earth's surface.

2. *Calculation of Unit Vectors*
The components of vectors **C**, **P**, and **G** can be derived from the Cartesian coordinates of points F, C, P, and G, and then converted to unit vectors by dividing each component by the magnitude of the vector, as follows.

Given the cartesian coordinates of two points (x_1, y_1, z_1), and (x_2, y_2, z_2), the vector **A** defining the length and direction of the line from point 1 to point 2 is given by the formula:

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \qquad (2)$$

where the vector components A_1, A_2, and A_3 are

$$A_1 = x_2 - x_1$$
$$A_2 = y_2 - y_1$$
$$A_3 = z_2 - z_1$$

and

i, **j**, **k** are unit vectors in the direction of the positive X, Y, Z axes respectively.

The components of the unit vector $\hat{\mathbf{A}}$ can be calculated by dividing each component A_1, A_2, A_3 of the vector **A** by the magnitude of the vector $|A|$.

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|A|} \qquad (3)$$

where

$$|A|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Thus:

$$\hat{\mathbf{C}} = \frac{\mathbf{C}}{|C|}, \quad \hat{\mathbf{P}} = \frac{\mathbf{P}}{|P|}, \quad \hat{\mathbf{G}} = \frac{\mathbf{G}}{|G|}.$$

3. *Calculation of angles β and θ*
The angles β and θ at the focal point are angles between vectors **C** and **P**, and vectors **C** and **G** respectively. They may be calculated by using the vector *dot product*, as follows:

For two vectors **A** and **B** the vector *dot product* is:

$$\mathbf{A}\cdot\mathbf{B} = |A|\,|B|\cos\psi \qquad (4)$$

where

ψ is the angle between **A** and **B**,
$|A|$ and $|B|$ are the magnitudes of **A** and **B**.

and since the magnitude of a unit vector is 1, then (4) may be simplified by using unit vectors, thus:

$$\hat{\mathbf{A}}\cdot\hat{\mathbf{B}} = \cos\psi \qquad (5)$$

The vector *dot product* is a scalar quantity S, and can be calculated from the vector components as:

$$S = A_1B_1 + A_2B_2 + A_3B_3 \qquad (6)$$

and hence, for unit vectors the angle between them is given by

$$\cos\psi = S \qquad (7)$$

In this way, the angles β and θ are given by:

$$\cos\theta = \hat{\mathbf{G}}\cdot\hat{\mathbf{C}}$$
$$\cos\beta = \hat{\mathbf{C}}\cdot\hat{\mathbf{P}}$$

4. *Calculation of the unit vector* $\hat{\mathbf{D}}$ *in the projection plane*
The unit vector $\hat{\mathbf{D}}$ in the direction of the line C′ P′ in the pro-

jection plane is calculated using vector *cross products*, as follows.

For two vectors **A** and **B**, the vector *cross product* is:

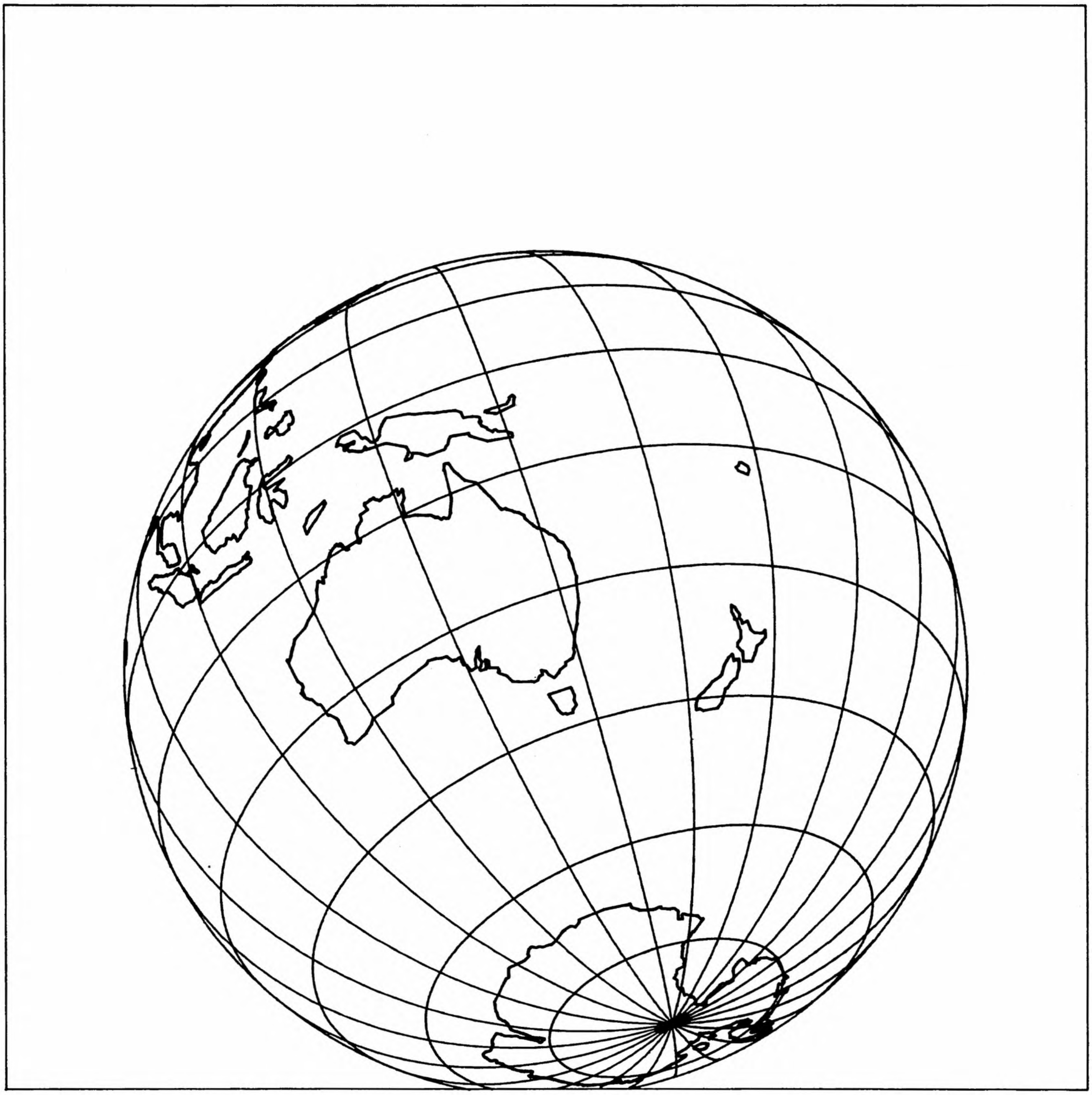
$$\mathbf{A} \times \mathbf{B} = |\mathrm{A}|\,|\mathrm{B}| \sin\psi\, \hat{\mathbf{U}} \qquad (8)$$

where

ψ is the angle between **A** and **B**,
$|\mathrm{A}|$ and $|\mathrm{B}|$ are the magnitudes of **A** and **B**,
$\hat{\mathbf{U}}$ is a unit vector perpendicular to the plane containing **A** and **B** and in the direction of a right-handed screw rotated from **A** to **B**.

Re-arranging (8) gives:

$$\hat{\mathbf{U}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathrm{A}|\,|\mathrm{B}| \sin\psi} \qquad (9)$$



TILTED CAMERA PERSPECTIVE PROJECTION

GRATICULE INTERVAL 15 DEGREE
CAMERA CENTRED AT (LAT,LONG) = −25,150
SATELLITE GROUND POINT (LAT,LONG) = −38,145
CAMERA FOCAL DISTANCE = 30.0 CMS
SATELLITE HEIGHT = 25,000 KMS

Figure 3. Perspective view of Australia and New Zealand from a height of 25,000km.

The result of a vector *cross product* (8) is another vector whose components are given by:

$$\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{i} - (A_1B_3 - A_3B_1)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k} \quad (10)$$

and the components of the unit vector $\hat{\mathbf{U}}$ are found by dividing each component of the *cross product* (10) by the magnitudes of vectors **A** and **B** and the sine of the angle between them, as in (9). If unit vectors are used, (9) is simplified, and if vectors **A** and **B** are perpendicular (9) is further simplified since $\sin(90°) = 1$.

TILTED CAMERA PERSPECTIVE PROJECTION

GRATICULE INTERVAL 15 DEGREE
CAMERA CENTRED AT (LAT,LONG) = −25,150
SATELLITE GROUND POINT (LAT,LONG) = −38,145
CAMERA FOCAL DISTANCE = 30.0 CMS
SATELLITE HEIGHT = 5,000 KMS

Figure 4. Perspective view of Australia from a height of 5,000km.

Using the relationships above, unit vector $\hat{\mathbf{D}}$ is obtained by the following vector *cross products*:

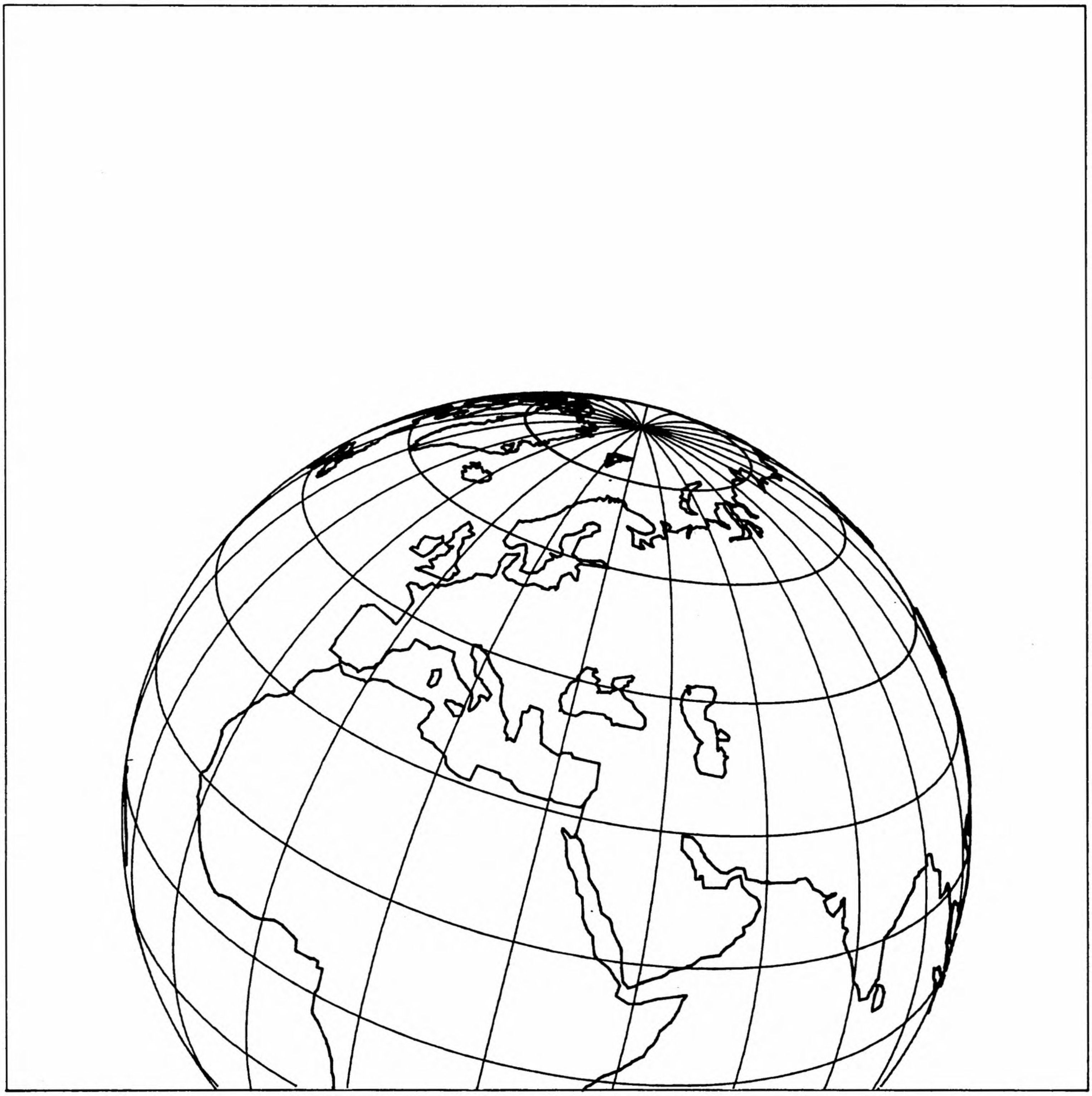
(i) $\hat{\mathbf{G}} \times \hat{\mathbf{C}} = \sin\theta\,\hat{\mathbf{X}}$ gives a unit vector $\hat{\mathbf{X}}$ perpendicular to the plane defined by vectors $\hat{\mathbf{G}}$ and $\hat{\mathbf{C}}$. Since the Y′ axis lies in the plane FGC, then $\hat{\mathbf{X}}$ is the unit vector in the direction of the X′ axis of the focal plane.

Thus:

$$\hat{\mathbf{X}} = \frac{\hat{\mathbf{G}} \times \hat{\mathbf{C}}}{\sin\theta}$$

(ii) Similarly, the unit vector in the direction of the Y′ axis is given by

$$\hat{\mathbf{Y}} = \hat{\mathbf{X}} \times \hat{\mathbf{C}}$$



TILTED CAMERA PERSPECTIVE PROJECTION

GRATICULE INTERVAL 15 DEGREE
CAMERA CENTRED AT (LAT,LONG) = 60,15
SATELLITE GROUND POINT (LAT,LONG) = 30,30
CAMERA FOCAL DISTANCE = 30.0 CMS
SATELLITE HEIGHT = 25,000 KMS

Figure 5. Perspective view of Europe, North Africa and the Middle East from a height of 25,000km.

since FC′ is perpendicular to the projection plane, and thus to the X′ axis.

(iii) A unit vector perpendicular to the FCP plane is given by

$$\hat{\mathbf{W}} = \frac{\hat{\mathbf{P}} \times \hat{\mathbf{C}}}{\sin\beta}$$

(iv) Thus, finally, the unit vector $\hat{\mathbf{D}}$ in the projection plane in the direction C′ P′ is given by

$$\hat{\mathbf{D}} = \hat{\mathbf{C}} \times \hat{\mathbf{W}}$$

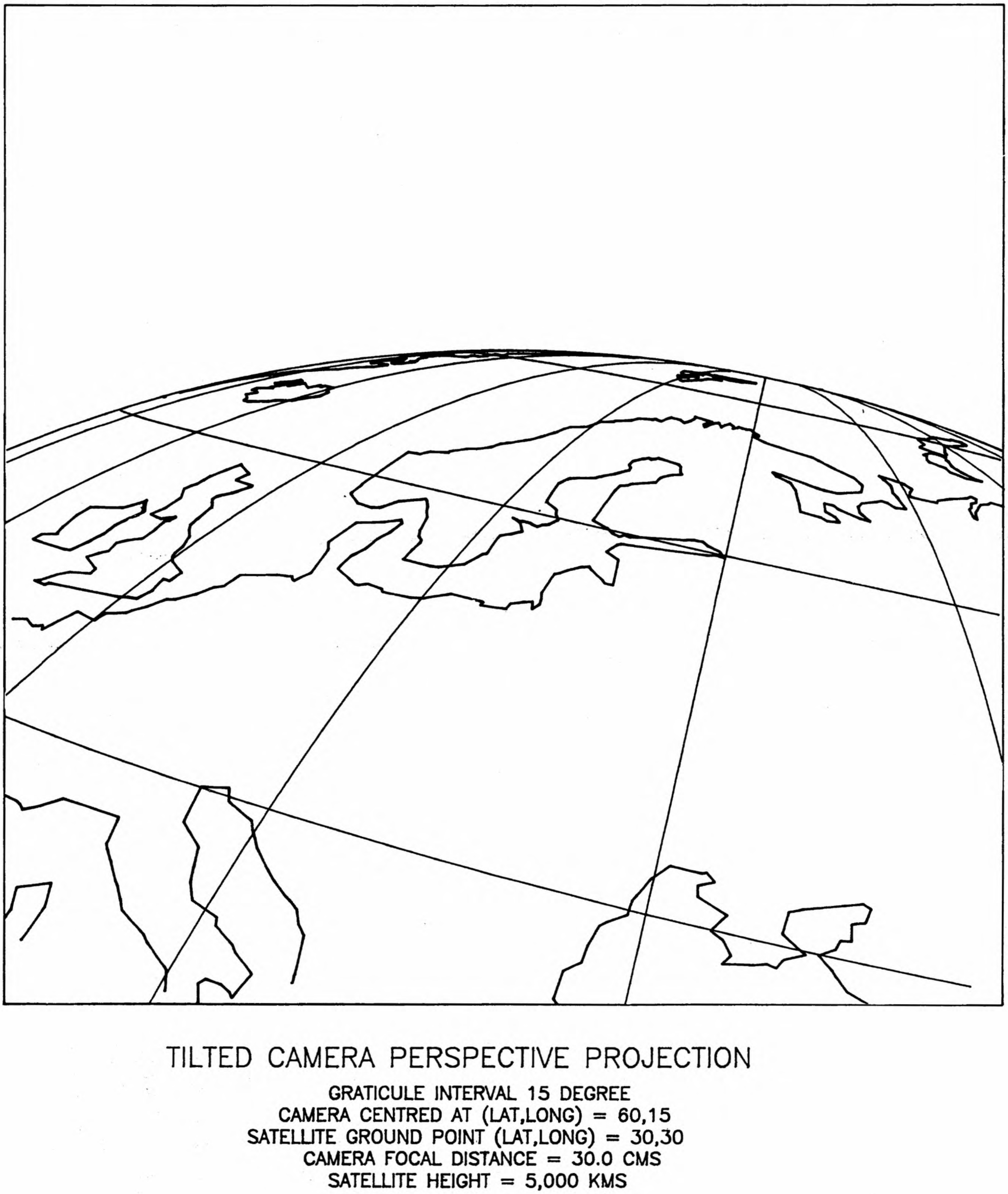


Figure 6. Perspective view of Europe and Scandinavia from a height of 5,000km.

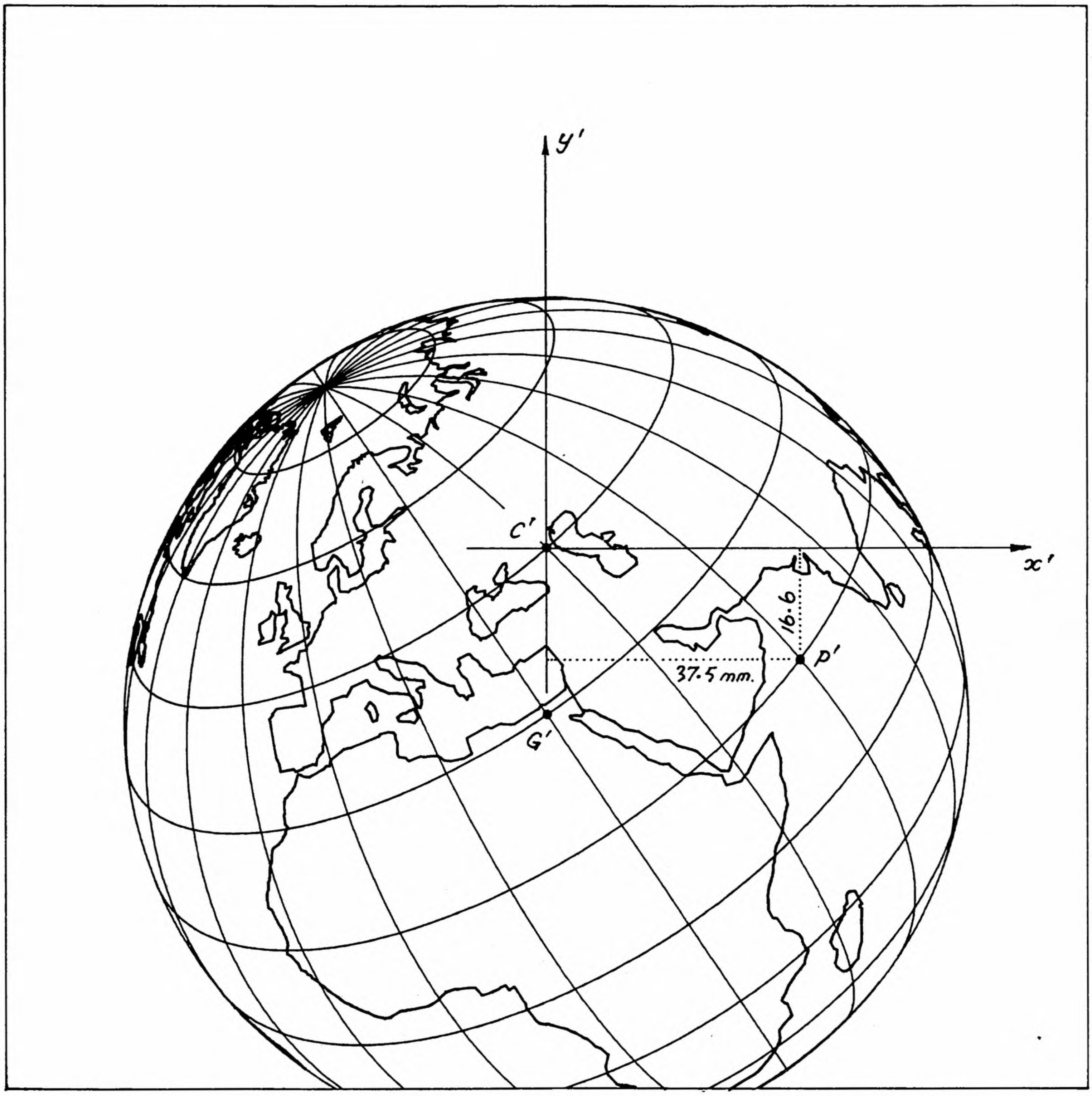
5. *The angle α in the projection plane*
The angle α, at the origin C′, measured clockwise from the Y′ axis to the projected point P′, can be deduced from the vector *dot products*

$$\cos\alpha = \hat{\mathbf{Y}}\cdot\hat{\mathbf{D}} \quad \text{and} \quad \cos\gamma = \hat{\mathbf{X}}\cdot\hat{\mathbf{D}}$$

and the rule

$$\text{if} \quad \gamma < 90° \quad \text{then} \quad \alpha = \alpha$$
$$\text{if} \quad \gamma > 90° \quad \text{then} \quad \alpha = 360° - \alpha$$

The angle γ is shown in *Figure 2* as the angle at the origin C′, measured clockwise from the X′ axis, and since the



TILTED CAMERA PERSPECTIVE PROJECTION

GRATICULE INTERVAL 15 DEGREE
CAMERA CENTRED AT (LAT,LONG) = 45,45
SATELLITE GROUND POINT (LAT,LONG) = 30,30
CAMERA FOCAL DISTANCE = 30.0 CMS
SATELLITE HEIGHT = 25,000 KMS

Figure A1. Perspective view of Europe, North Africa and the Middle East from a height of 25,000km. The 'space camera' is directly above point G′ and is pointed at point C′.

inverse cosine function will yield angular values between 0° and 180°, inspection of γ will determine the correct quadrant for α.

6. *The distance C′P′ in the projection plane*
If the distance FC′ is known then:

$$d = f \tan\beta$$

where:

f is the distance FC′
d is the distance C′ P′

7. *The projection plane coordinates of P′*
Finally, the projection plane coordinates of the point P′ are computed from simple trigonometry and have the same units of length as those chosen for the distance f.

Thus:

$$x' = d \sin\alpha$$
$$y' = d \cos\alpha$$

The projection plane coordinates (x′, y′), of complementary points (ϕ, λ) on the Earth's surface, having been calculated by the method set out above, can be used to construct a perspective projection of the Earth.

CONCLUSION

The method of computing projection coordinates outlined above has a wonderful simplicity involving nothing more than simple trigonometry and arithmetic and lends itself to computer programming. Program subroutines to calculate Cartesian coordinates given geographical coordinates, vector components given Cartesian coordinates, and the vector *dot* and *cross* products can be accessed sequentially within a computer program to convert geographical coordinates to projection coordinates.

This vector technique, with very little modification, can easily be extended to projections of the *spheroid*, and can accommodate points above or below the Earth's surface.

The theory of vectors and their manipulations can be found in many mathematical texts and handbooks, such as *Mathematical Handbook of Formula and Tables* (Spiegel, 1968).

Examples of projections of the spherical Earth using the method above are shown in Figures 3 to 6, and a worked example is contained in the following Appendix.

APPENDIX

Worked Example – *Tilted camera perspective projection*
Referring to *Figures 1* and *A1*, a 'space camera' is situated 25,000kms above G(ϕ_G = 30°N, λ_G = 30°E) on the Earth's surface and is pointed at C(ϕ_C = 45°N, λ_C = 45°E). The focal length of the 'space camera' is 30cms.

It is required to calculate the Cartesian coordinates of the projected point P(ϕ_P = 15°N, λ_P = 60°E).

In *Figure A1* the projection of points G, C, and P on the Earth's surface are shown as G′, C′, and P′.

1. *Calculation of Cartesian coordinates (X, Y, Z)*
Using equations (1) with the radius of the spherical Earth and the space camera flying height as:

$$R = 6{,}371\text{km} \quad \text{and} \quad H = 25{,}000\text{km respectively}$$

gives the following coordinates in kilometres.

$X_G = 4{,}778.2500$ $\quad X_C = 3{,}185.5000$ $\quad X_P = 3{,}076.9567$
$Y_G = 2{,}758.7239$ $\quad Y_C = 3{,}185.5000$ $\quad Y_P = 5{,}329.4454$
$Z_G = 3{,}185.5000$ $\quad Z_C = 4{,}504.9773$ $\quad Z_P = 1{,}648.93611$

$X_F = 23{,}528.2500$
$Y_F = 13{,}584.0415$
$Z_F = 15{,}685.5000$

2. *Calculation of unit vectors*
Using equations (2) the components of vector **C** are:

$$\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$$

where

$$C_1 = X_C - X_F = -20{,}342.7500\text{km}$$
$$C_2 = Y_C - Y_F = -10{,}398.5415$$
$$C_3 = Z_C - Z_F = -11{,}180.5227$$

and the magnitude of vector **C** is

$$|C| = 25{,}435.4326\text{km}$$

and using equation (3) the unit vectors $\hat{\mathbf{C}}$ $\hat{\mathbf{G}}$ and $\hat{\mathbf{P}}$ are

$$\hat{\mathbf{C}} = -0.7998\mathbf{i} - 0.4088\mathbf{j} - 0.4396\mathbf{k}$$
$$\hat{\mathbf{G}} = -0.7500\mathbf{i} - 0.4330\mathbf{j} - 0.5000\mathbf{k}$$
$$\hat{\mathbf{P}} = -0.7823\mathbf{i} - 0.3158\mathbf{j} - 0.5369\mathbf{k}$$

3. *Calculation of angles β and θ*
Using equations (6) and (7) the angles β and θ are derived from the unit vector components as

$$\cos\theta = \hat{\mathbf{G}}\cdot\hat{\mathbf{C}} = 0.9967 \quad \text{and} \quad \theta = 4°\ 41'\ 02''$$
$$\cos\beta = \hat{\mathbf{C}}\cdot\hat{\mathbf{P}} = 0.9908 \quad \text{and} \quad \beta = 7°\ 46'\ 35''$$

4. *Calculation of the unit vectors* $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$, *and* $\hat{\mathbf{D}}$ *in the projection plane*
Using equations (9) and (10) and referring to section 4, parts (i) to (iv), the following unit vectors are obtained

$$\hat{\mathbf{X}} = -0.1721\mathbf{i} + 0.8597\mathbf{j} - 0.4863\mathbf{k}$$
$$\hat{\mathbf{Y}} = -0.5767\mathbf{i} + 0.3133\mathbf{j} + 0.7579\mathbf{k}$$
$$\hat{\mathbf{W}} = -0.5961\mathbf{i} + 0.6320\mathbf{j} + 0.4968\mathbf{k}$$
$$\hat{\mathbf{D}} = +0.0747\mathbf{i} + 0.6594\mathbf{j} - 0.7492\mathbf{k}$$

5. *The angle α in the projection plane*
Referring to section 5 the angles α and γ are given by

$$\cos\gamma = \hat{\mathbf{X}}\cdot\hat{\mathbf{D}} \quad \text{and} \quad \gamma = 23°\ 18'\ 42''$$
$$\cos\alpha = \hat{\mathbf{Y}}\cdot\hat{\mathbf{D}} \quad \text{and} \quad \alpha = 113°\ 50'\ 52''$$

(Note that these angles should differ by 90° but due to rounding errors are slightly in error.)

6. *The distance C′P′ in the projection plane*
Referring to section 6.

$$d = 4.0969\text{cm}$$

7. *The projection plane coordinates of P′*

$$x' = d\ \sin\alpha = 3.75\text{cm}$$
$$y' = d\ \cos\alpha = -1.66\text{cm}$$

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Snyder, J. P., 1981. *The Perspective Map Projection of the Earth.* The American Cartographer, 8 (2): 149-160.
Spiegel, M. R., 1980. *Schaum's Outline Series – Mathematical Handbook of Formulas and Tables.* McGraw-Hill, New York.